Vlasov Eqn. → char. orbits
→ Jeans eqns.

I.) Orbital Mechanics → Disk Potential

A.) Epicyclic Approximation (Linear)

Have:

→ disk galaxy

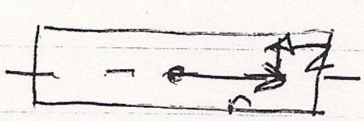
→ cylindrical symmetry ⇒ $V = V(r, z)$

Seek:

→ basic characterization of stellar orbits in (azimuthally) symmetric potential (e.g. equilibrium radius, small oscillation frequency, etc.)

→ determine integrability vs. "approximate" integrability vs. stochasticity of orbits for different initial conditions

Now,



$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - V(r, z)$$

$$m\ddot{r} - mr\dot{\phi}^2 = -\frac{\partial V}{\partial r} \quad ; \quad m\ddot{z} = -\frac{\partial V}{\partial z}$$

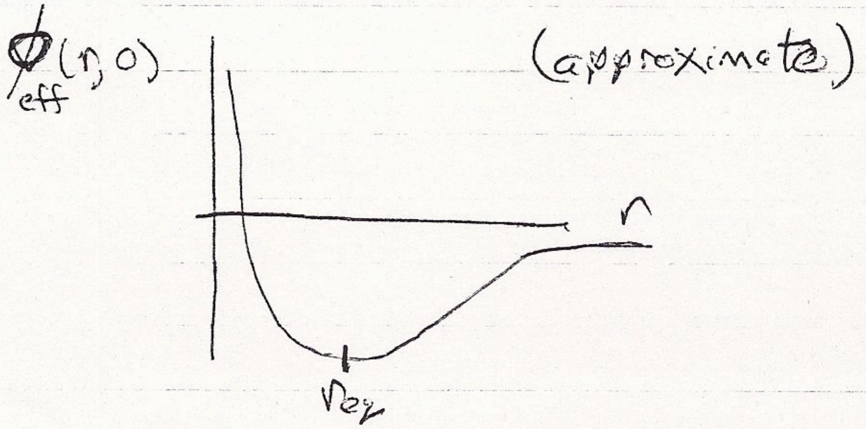
$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0$$

$$\dot{\phi} = \frac{L_z}{mr^2}$$

$\Rightarrow$

$$m\ddot{r} = mr\frac{L_z^2}{m^2r^4} - \frac{\partial V}{\partial r} \quad ; \quad m\ddot{z} = -\frac{\partial V}{\partial z}$$

$$\equiv -\frac{\partial}{\partial r}\underbrace{\left(V(r,z) + \frac{L_z^2}{2mr^2}\right)}_{\text{effective potential}} \equiv -\frac{\partial}{\partial r}\Phi_{eff.}$$



$\therefore$ first, seek expand about $r = r_{eq}$, $z = 0$ (i.e. $\frac{\partial V}{\partial z} = 0$ at $z=0$ by Gauss's Law)

$r_{eq} \Rightarrow \partial \Phi_{eff}/\partial r = 0$

$$+\frac{L_z^2}{mr^3} = \frac{\partial V}{\partial r} \Rightarrow \text{gives } r_{eq}.$$

Observe: $L_z = m r_{eq}^2 \underbrace{\Omega_c(r_{eq})}_{\text{"circular frequency"}}$

$$\underbrace{\Omega_c^2}_{\text{"circular frequency"}}(r_{eq}) = \frac{m\, r_{eq}^3}{m^2 r_{eq}^4}\left.\frac{\partial V}{\partial r}\right|_{r_{eq},\, z=0}$$

$$\therefore \quad \Omega_c^2 = \frac{1}{m r_{eq}} \left.\frac{\partial V}{\partial r}\right|_{r_{eq},0}$$ → defines circular frequency at r_{eq}

Now, for small oscillations about $r = r_{eq}$, $z = 0$;

$$\phi_{eff} = \underbrace{\phi_{eff}(r_{eq},0)}_{\text{ign. const.}} + \cancelto{0}{z \frac{\partial \phi_{eff}}{\partial z}} + \cancelto{0}{(r - r_{eq}) \frac{\partial \phi_{eff}}{\partial r}}$$

$$+ \frac{1}{2}(r - r_{eq})^2 \left.\frac{\partial^2 \phi_{eff}}{\partial r^2}\right|_{r_{eq},0} + \frac{1}{2} z^2 \left.\frac{\partial^2 \phi_{eff}}{\partial z^2}\right|_{r_{eq},0} + \cancelto{0}{\left.\frac{\partial^2 \phi}{\partial r \partial z}\right|_{r_{eq},0} (r - r_{eq}) z}$$

$$\phi_{eff} = \frac{1}{2} x^2 \left.\frac{\partial^2 \phi_{eff}}{\partial r^2}\right|_{r_{eq},0} + \frac{1}{2} z^2 \left.\frac{\partial^2 \phi_{eff}}{\partial z^2}\right|_{r_{eq},0} \qquad x \equiv r - r_{eq}$$

$$\therefore \quad H = \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) + \phi_{eff}(x, z)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) + \frac{1}{2}\left(x^2 \phi_{xx} + z^2 \phi_{zz}\right)$$

→ harmonic oscillator

↓

key to all physics.

Now, $$\left.\frac{\partial^2 \phi_{eff}}{\partial r^2}\right|_{r_{eq}} = \left.\frac{\partial^2 V}{\partial r^2}\right|_{r_{eq},\, z=0} + \frac{3 L_z^2}{\cancel{2} m r_{eq}^4}$$

but $\left.\partial V / \partial r\right|_{r_{eq}} = L_z^2 / m r_{eq}^3$; $L_z = m r^2 \Omega_c$

$$\therefore \quad \Omega_o^2 = \frac{1}{r}\frac{\partial V}{\partial r} = \frac{L_z^2}{r^4} \qquad (m \to 1, \text{ as cancels}).$$

$\Rightarrow$

r, z oscillators:

i) $\ddot{x} + \kappa^2 x = 0$

$$\kappa^2 = \left.\frac{\partial^2 V}{\partial r^2}\right|_{r_{eq}} + \frac{3L_z^2}{r_{eq}^4} = \left[\frac{\partial}{\partial r}\left(r\Omega_o^2\right) + 3\Omega_o^2\right]_{r_{eq}}$$

(hereafter drop 'sub-o')

$$= \left(4\Omega^2 + 2r\frac{\partial}{\partial r}\Omega^2\right)_{r_{eq},\, z=0}$$

$$= \left.\frac{2\Omega}{r}\frac{\partial}{\partial r}\left(r^2\Omega\right)\right|_{r_{eq}}$$

and

ii) $\ddot{z} + \nu^2 z = 0$

$$\nu^2 = \frac{\partial^2 \phi_{eff}}{\partial z^2}$$

Very important

Now

$$- \boxed{\kappa^2 = 4\Omega^2 + 2r\frac{\partial}{\partial r}\left(\Omega^2\right)} \rightarrow \boxed{\text{epicyclic frequency}}$$

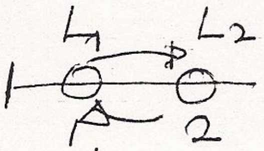
oscillation frequency } small radial excursions from r_{eq}

observe $\kappa^2 = \frac{2\Omega}{r}\left(\frac{\partial}{\partial r}(r^2\Omega)\right)_{r_{eq}}$

$\hookrightarrow$ {slope of angular momentum profile!

i.e. $\begin{cases} \frac{\partial}{\partial r}(r^2\Omega) > 0 & \Rightarrow \text{stable (epicyclic oscillations) profiles.} \\ \frac{\partial}{\partial r}(r^2\Omega) < 0 & \Rightarrow \text{unstable (to epicycles) profiles.} \end{cases}$

i.e. consider ΔE for interchange of 2 rings:



(angular momentum conserved) → {$m=0$ rings

$$E_{before} = \frac{L_1^2}{r_1^2} + \frac{L_2^2}{r_2^2}$$

$$E_{after} = \frac{L_2^2}{r_1^2} + \frac{L_1^2}{r_2^2}$$

$\boxed{\text{Rayleigh criterion}}$

$\begin{cases} \Delta E > 0 \rightarrow \partial(r^2\Omega)/\partial r > 0 \\ \Delta E < 0 \rightarrow \partial(r^2\Omega)/\partial r < 0 \end{cases}$

- Continuum Picture $\leftrightarrow$ Couette Flow $(m=0)$
  $\hookrightarrow$ cent. force

$$\frac{\partial \tilde{V}_r}{\partial t} + \frac{\tilde{V}_\theta^2}{r} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial r}$$

$$\frac{\partial \tilde{V}_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial z}$$

$\hookrightarrow$ {Can / Does κ^2 (epi-freq.) appear in continuum picture (i.e. rotating fluid)

$m=0$

$$\frac{\partial}{\partial t}\tilde{V}_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega)\tilde{V}_r - \frac{1}{\rho_0}\cancel{\nabla_\theta \tilde{P}_0}$$

$\underline{\nabla}\cdot\underline{V} = 0$

$\Rightarrow$ write $\tilde{\underline{V}} = (\tilde{V}_r, \tilde{V}_z)$

$$\underline{\nabla}\cdot\underline{V} = 0 \Rightarrow \tilde{\underline{V}} = \nabla\phi \times \vec{y}$$

$$\therefore \quad V_r = -\partial_z\phi$$

$$V_z = \partial_r\phi$$

$$\underline{\omega} = \underline{\nabla}\times\underline{V} \; ; \quad \underbrace{\omega_y}_{\text{vorticity component in } \vec{y} \text{ direction}} = -\partial_z^2\phi - \partial_r^2\phi = -\nabla_\perp^2\phi$$

so

$$\frac{\partial}{\partial t}\left(-\nabla_\perp^2\tilde{\phi}\right) = -2\Omega\,\partial_z\tilde{V}_\theta$$

$$\frac{\partial}{\partial t}\tilde{V}_\theta = -\partial_z\tilde{\phi}\,\frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega)$$

$\ast$

$\Rightarrow$

$$-i\omega\left(+k_\perp^2\tilde{\phi}_{k,\omega}\right) = -2\Omega\, ik_z\, V_{\theta_{k,\omega}}$$

$$-\cancel{i}\omega\tilde{V}_{\theta_{k,\omega}} = -\cancel{i}k_z\frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega)\tilde{\phi}_{k,\omega}$$

$$k_\perp^2 = k_z^2 + k_r^2$$

$$\omega^2 k_\perp^2 = k_z^2 \frac{2\Omega}{r}\frac{\partial}{\partial r} r^2\Omega$$

$$\boxed{\omega^2 = \frac{k_z^2}{k_\perp^2}\Phi\,, \quad \Phi = \frac{2\Omega}{r}\frac{\partial}{\partial r}\left(r^2\Omega\right)}$$

Couette Flow:
Radial Buoyancy wave
Dispersion Relation

Rayleigh Discriminant
$\Phi > 0$, $(r^2\Omega)' > 0 \Rightarrow$ stable

$\Phi < 0$, $(r^2\Omega)' < 0 \Rightarrow$ unstable
(Rayleigh criterion)

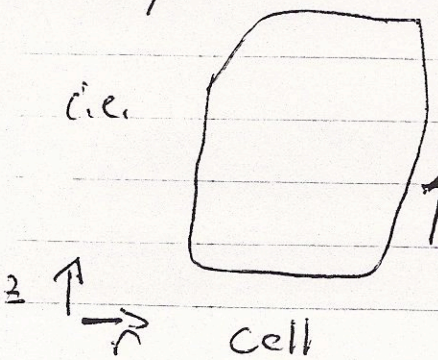
Observe:

- Buoyancy Wave: $\omega^2 = \dfrac{k_z^2}{k^2}\boxed{\kappa^2}$ $\Rightarrow$ frequency is epicyclic, with k_z^2/k^2 factor

i.e. epicyclic frequency appears in continuum picture as ~ buoyancy wave frequency (clear analogy to small oscillation frequency of particle)

$k_z^2/k^2 \Rightarrow$ incompressibility of r, z cell

i.e.



$\leftarrow F_{restoring}\ \hat{r}$

$\uparrow \nabla_z P$ maintains incompressibility
(i.e. force driving vertical motion vs. inertia)

$\therefore$ for long, thin cells, takes longer for vertical motion to maintain $\nabla \cdot \underline{V} = 0$ $\Rightarrow$ frequency must drop with k_z $\Rightarrow k_z^2/k^2$.

(Preview: $k_\theta = 0$. dens. wave: $\omega^2 = \Phi + k_r^2 c_s^2 - 2\pi G \sigma_0 |k_r|$)

- $\omega \to \omega + i\nu k^2$ recovers Taylor # criterion for $\kappa^2 < 0$ instability

$\Rightarrow$ Relation to Oort Parameters A, B:

Oort constants give local speed curve via local measurement of stellar motion

$$A = \frac{1}{2}\left(\frac{V_c}{r} - \frac{dV_c}{dr}\right)_{R_0} \equiv \text{measure of net shear in galactic motion (locally)}$$

$$= -\frac{1}{2}\left(r\frac{\partial \Omega}{\partial r}\right) \quad \text{shear}$$

shear

$V = \Omega_0 r$, a/c solid body $\Rightarrow A = 0$

Ω_0: const.

$$B = -\frac{1}{2}\left(\frac{V_c}{r} + \frac{dV_c}{dr}\right) = -\frac{1}{2r}\frac{d}{dr}(rV_c) \equiv \text{angular momentum gradient local.}$$

$$= -\left(\frac{1}{2}r\frac{\partial \Omega}{\partial r} + \Omega\right)$$

vorticity

since $\kappa^2 = \left(R\frac{\partial \Omega^2}{\partial r} + 4\Omega^2\right)_{r_{eq}}$

$\Rightarrow$ $\kappa^2 = -4B(A-B) = -4B\Omega(r_{eq})$

$\Omega(r_{eq}) = A - B$

-- Physical Picture of Epicyclic orbits

- epicycle is radial (in-and-out) oscillation (of star within disk; executing quasi-circular orbit) concurrent with orbit about galactic center

i.e. for sun $\dfrac{\kappa}{\Omega_0} = 2\left(\dfrac{-B}{A-B}\right)^{1/2} \cong 1.3$, for sun

- clearly, azimuthal oscillation accompanies radial

i.e.

$$m r^2 \dot{\phi} = L_z$$

$\downarrow$ C.o.M.

{ note: in $z=0$ plane, problem equivalent to planetary orbit

$$r^2 = (r_0 + x)^2 \cancel{+ z^2}$$

in disk plane (z {harmonic oscillation} for $z \lesssim 300\,\mathrm{psc.}$)

$$\Rightarrow \quad \dot{\phi} = L_z / m(r_0+x)^2 = L_z / m r_0^2 (1 + x/r_0)^2$$
$$\cong \frac{L_z}{m r_0^2}\left(1 - \frac{2x}{r_0} + \ldots\right)$$

but $x = x_0 \cos(\kappa t + \alpha)$

$$\left\{ \begin{aligned} \phi &= \frac{L_z}{m r_0^2}\left(t - \frac{2x_0}{r_0 \kappa}\sin(\kappa t + \alpha)\right) \cancel{+ \phi_0} \\ &= \Omega_0 \left(t - 2x_0/r_0\kappa \sin(\kappa t + \alpha)\right) \cancel{+ \phi_0} \\ r &= r_0 + x_0 \cos(\kappa t + \alpha) \end{aligned} \right.$$

guiding center (epicenter) $\begin{cases} r = r_0 \\ \phi = \Omega_c t + \phi_0 \end{cases}$ i.e. $\begin{cases}\text{circular} \\ \text{orbit}\end{cases}$

Now, note (residual) epicyclic motion elliptical:

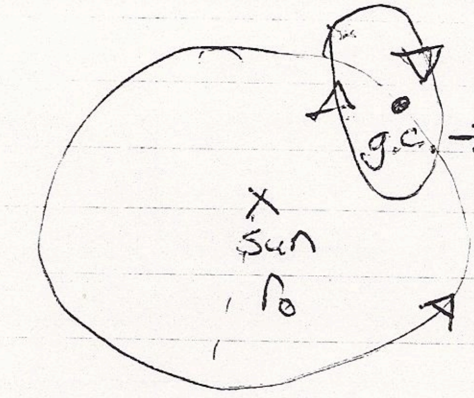
$$\begin{cases} x = x_0 \cos(\kappa t + \alpha) \\ y = r_0(\phi - \Omega_c t) = -\dfrac{2x_0\Omega_c}{\kappa} \sin(\kappa t + \alpha) \end{cases}$$

$\dfrac{x}{y} = \dfrac{\kappa}{2\Omega_c}$; motion is retrograde (i.e. $y \sim -x_0(\sin \kappa t + \alpha)$)

$= 1/2$, for Kepler (i.e. sun + planet)

[semi-axial ratio]

Thus, net motion of planet $=$ $\begin{Bmatrix}\text{circle of} \\ \text{guiding} \\ \text{center}\end{Bmatrix}$ $+$ $\begin{Bmatrix}\text{ellipse} \\ \text{of} \\ \text{epicycle}\end{Bmatrix}$ (retrograde)



$\rightsquigarrow x/y = 1/2 \Rightarrow$ ellipse elongated in azimuthal direction

$\to \phi = \Omega_c t$

(opposite sense)

i.e. elliptical Kepler orbit $\cong$ $\begin{Bmatrix}\text{prograde} \\ \text{circular} \\ \text{guiding center}\end{Bmatrix}$ $+$

$\begin{Bmatrix}\text{retrograde epicyclic ellipse.} \\ \text{(elongated tangentially)}\end{Bmatrix}$

- rev
- why 3rd
- Arnold – law

- Ulrsoo
- Epi

B.) Problem of the "Third Integral"

→ recall have been discussing $V = V(r, z)$ potential, and have $(E/m) = \text{const}$,

$$E = \frac{1}{2}(\dot{r}^2 + \dot{z}^2) + \frac{L_z^2}{2r^2} + V(r, z)$$

so clearly have Hamiltonian:

$$H = \frac{1}{2}\left(p_r^2 + p_z^2\right) + \frac{L_z^2}{2r^2} + V(r, z)$$

trivial
E, L_z → axisymm
but $U = U(r, z)$?!
① if $U = U(r) + V(z)$ → HO.
3rd integral → E_z, i.e. E_r, E_z
② sphere L_z, $L^2, |L|$,
= → surface of section

star stellar motion clearly has $\underline{\underline{2}}$ integrals of motion E, L_z, at least!

→ $\underline{\underline{\text{But}}}$, have observation:

→ stellar velocities in meridian plane (r, z) have preferred distribution (obs.)

→ if E, L_z are only I.O.M.'s, $f((v_r, v_z), r)$ should be uniform in meridian plane, as $f = f(\text{IOM's})$, only! – a contradiction!!
(n.b.: time scale –) i.e. ergodicity

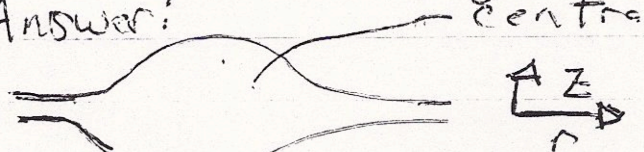
∴ → there must exist a THIRD I.O.M. in addition to L_z, E!!! → or why not uniform?

$\boxed{\text{The Problem}}$: How relate '$3^{rd}$ integral' to p_r, p_z, r, z etc. → i.e. what is it? if it is…

⇒ { Henon-Heiles Potential
Program of Contopoulos, et. al.

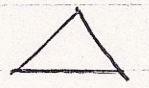
1.) Henon-Heiles Potential

→ what does a galaxy look like, meridionally (i.e. from side)?
Answer:



⇒ V must have hard inner wall → centrifugal eff potential; but stars must escape at large distance ($F \sim GMm/r^2$!)

H-H potential : V = const. along sides of equilateral triangle (symmetry ⟷ simplicity)

(i.e.  →  !?)

$(r, z \to x, y)$

so,

$$V(x,y) = (y + 1/2)\left(x^2 - (y-1)^2/3\right)$$

$= 0$ for $\begin{cases} y = -1/2 \\ x = (y-1)/\sqrt{3} \\ x = -(y-1)/\sqrt{3} \end{cases}$

unrel.

$$\Rightarrow \quad V(x,y) = \frac{x^2+y^2}{2} + x^2 y - \frac{y^3}{3} - 1/6$$

so H-H Hamiltonian given by:

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2) + \lambda\left(x^2 y - \frac{y^3}{3}\right)$$

where: $\lambda = m\omega^2/a$

some length (shape related)

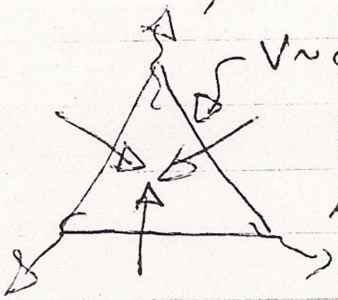
m, ω, $a = m\omega^2/\lambda$ } 3 consts of phys system

observe:

- spherical symmetry $\rightarrow$ no length scale in V

$\Rightarrow$ meridional structure $\rightarrow$ V must contain scale info $\leftrightarrow$ origin of λ ....

- $V(x,y) = (y + 1/2)\left(x^2 - (y-1)^2/3\right)$



strong attraction $\perp$ equilateral sides

leakage possible due weak attraction at corners (i.e. $V<0$ at $x=0$)

$\Rightarrow$ consistent with meridional asymmetry.

{ In Search of the Third Invariant:
→ { What do Orbits in H-H Potential Look Like?

i.e. { - clues
 - approximate invariant

- Surface of Section Description

Have 4D system $\left\{ \begin{matrix} p_r, p_z, r, z \\ p_x, p_y, x, y \end{matrix} \right\}$ ⇒ too difficult to visualize!

∴ try depict reduced phase space ↔ exploit IOM's

- $E(r, z, \dot{r}, \dot{z}) = $ const. ⇒ plot $r, z, \dot{r}$
(eliminating $\dot{z}$ via E)
→ still too difficult to visualize

- plot $r, \dot{r}$ coordinates (2D phase space - all we can draw!) of star/particle crossing $z = \text{const} = 0$ plane.

- remove $\dot{z}$ ambiguity from plot via $\dot{z} > 0$, only plotted

∴ "Surface of section" ≡ $\dot{z} > 0$ trajectory "puncture plot" of $r, \dot{r}$ plane, for $z = 0$

⇒ reduces 4D representation to 2D

⇒ can vary $z = $ const. → different "surfaces of section"

→ Using Surface of Section

- choose : $x = 0$ { for surface of section (symmetry of triangle!)

  plot : $\dot{x} \geq 0$ crossings, only

  for E, given

⇒ puncture points have $\frac{p_y^2}{2} + V(0,y) < E$,

which specifies domain of surface of section (boundary) $D(E)$

- initial point arbitrary $P_0 = (p_{y0}, y_0)$ ⇒ subsequent $P_1 = (p_{y1}, y_1)$, $P_2 = (p_{y2}, y_2)$, ⋮

with $x = 0$, $\dot{x} > 0$.

- result:

qualitatively, either

- puncture points line up on smooth curve
  → trajectory on torus within energy surface
  ⇒ "3rd invariant"

or

- puncture points not accomodated on smooth curve
  ⇒ no invariant torus
  ⇒ no additional invariant

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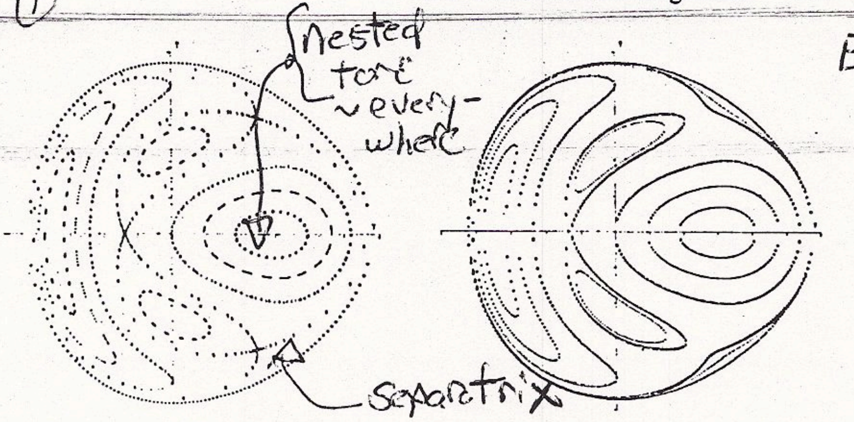
1

E = 1/12

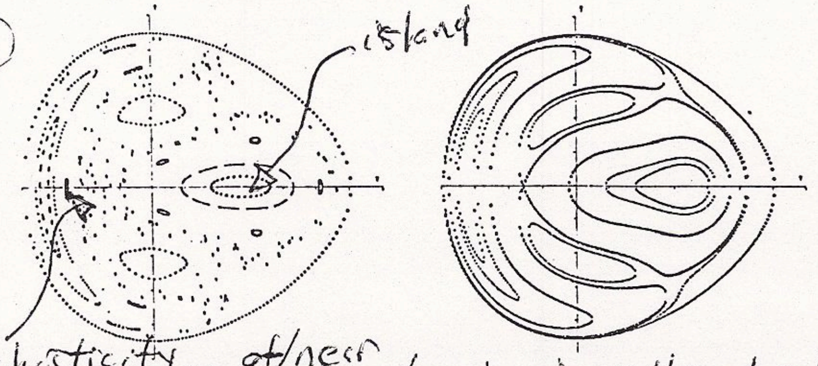
lines shape indicate same trajectory



Figures 12a and 12b Surfaces of section for the Hénon-Heiles potential at the energy 1/12, from numerical integration (a), and from Birkhoff-Gustavson renormalization (b) [from Gustavson (1966)].

2

E = 1/8



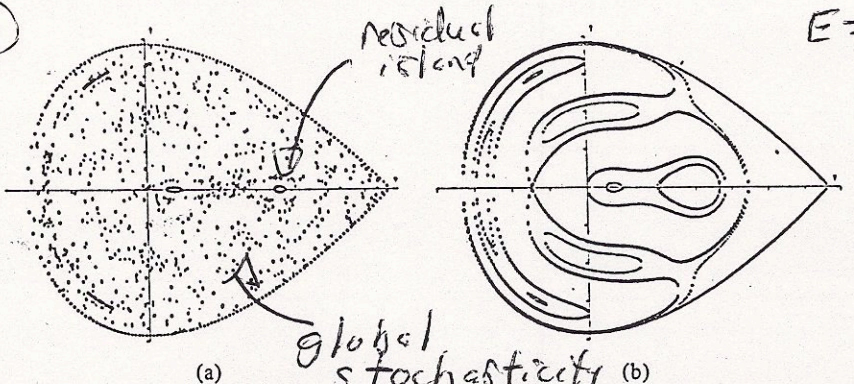
stochasticity at/near x-point → connection/wandering

three tori

Figures 13a and 13b Same as preceding figure for the energy 1/8.

3

E = 1/6



Figures 14a and 14b Same as preceding two figures for the escape energy 1/6.

Note:

→

1. - nested tori @ everywhere ⇒ nearly integrable, with approximate 3rd IOM

   - separatrix evident → cf. vibration - libration boundary for pendulum

   - separatrix intersects self ⇒ x-point

$E = 1/12 \ll E_{escape}$ from center of triangle

2. - ergodicity evident ⇒ scatter of <u>single</u> trajectory evident!
   - ergodicity starts at x-points (basins meet mountains)

   - islands (remaining closed loops) correspond to tori in 1

3. - global stochasticity, albeit only $y=1, \dot{y}=0$ can escape
   - tiny residual islands

→ H-H system is equivalent to <u>truncated</u> Toda lattice, which is <u>non-integrable</u>.

Observe:

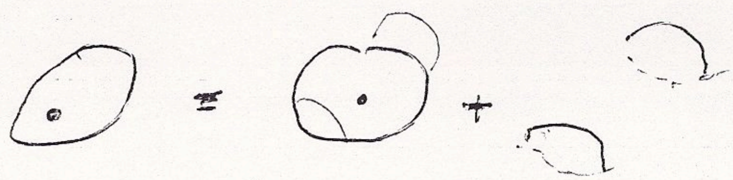
For Keplerian orbit: $r\Omega^2 = \frac{GM}{r^2}$

$$\Omega = \Omega_0 \, (r_0/r)^{3/2}$$

$$\kappa^2 = \frac{2\Omega}{r}\,(r^2\Omega) = \Omega^2$$

$\Rightarrow$

$$\kappa^2/\Omega^2 = 1$$

- 1 epicycle per year, for earth!
- 

→ what's the Implication for Galaxy ?

- stellar motion in $V(r, z)$ likely non-integrable, possibly stochastic (depends on parameters)

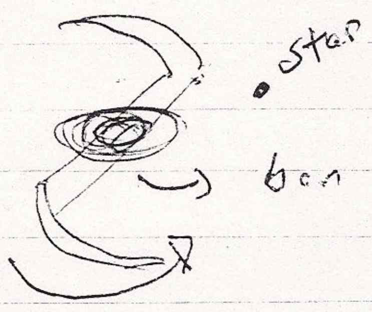
- escape from Galaxy possible

⇒ integration of Vlasov eqn. !? → resonance broadening

c.) Rotating Potential - Bars, etc. $\left\{ \begin{array}{l} \text{Co-rotation} \\ \text{Lindblad} \end{array} \right.$ Resonances.

- in general, galaxies can $\left\{ \begin{array}{l} \text{rotate} \\ \text{non-axisymmetric} \end{array} \right.$

i.e. barred galaxies (B. & T. pg. 400)



clearly non-axisymmetric structure

where bar/spiral structure rotates.

- thus, individual star 'feels' time dependent, non-axisymmetric potential

- what happens ? → new frequency appears, namely Ω_b - (bar) rotation frequency orbits symmetry — 3rd integral theory (1)

$\rightarrow$ rev. — exp $\left(r \frac{d(\Omega^2)}{dr} + 4\Omega^2 \right) = \frac{2r\Omega\, d\Omega + 4\Omega^2}{dr}$

$= \frac{2\Omega}{r} \frac{d}{dr}(r^2 \Omega)$

$= \frac{2\Omega}{r}\left(2r\Omega + r^2 \frac{d\Omega}{dr}\right)$

$= 4\Omega^2 + r\frac{d}{dr}(\Omega^2)$

(A) rev. [illegible]

(B) Bar $\rightarrow$ [illegible]

(C) Trapping

— asymmetry — bar

— [illegible]

— co-rotation

$m(\Omega_p - \Omega_b) = \pm \kappa$

— resonance $\rightarrow$

— Co-rotation — closes — trapping

— static frame ref. — eq

$\partial \phi_{eq} = 0$ — stab. study

— look at Lagrange pts

Jacobi int

— orbits trapped (stable)

theory

— time evol. @ δr E

a ?

[illegible] effect

Donkey Effect $\rightarrow$ how do stars respond to tang. external force

— sheep $\rightarrow$ go in direction (normal) perfectly elastic

— donkey $\rightarrow$ reverse $\rightarrow$ if pulled/pushed slows, etc.

$\phi'' = \frac{d\Omega}{dr}\dot{r} = \frac{\Omega' r F}{m(2\Omega r + r^2\Omega')}$ $\rightarrow = \frac{\Omega' F}{2m\left(\Omega + \frac{r\Omega'}{2}\right)}$

$L = r^2\Omega$

$\dot{L} = (2r\Omega + r^2\Omega')\dot{r}$

$\dot{L} = rF/M$ $\quad \frac{rF}{m} = (2r\Omega + r^2\Omega')\dot{r}$

$A = -\frac{1}{2} r \, d\Omega/dr > 0$

$B = -\left(\Omega + \frac{1}{2} r \frac{d\Omega}{dr}\right) < 0$

OORT

?

$\alpha + 2$

$$\ddot{\phi} = -\frac{\Omega' F}{2m\left[-\left(\Omega + \frac{r\Omega'}{2}\right)\right]}$$

~~BA~~

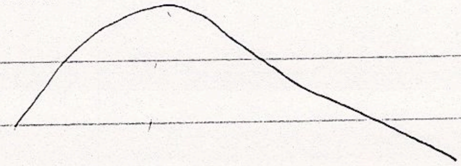
$$= \frac{A}{2mrB} F$$

$$\boxed{\ddot{\phi} = \frac{A}{2mrB} F}$$

$$\frac{A}{B} = \frac{\frac{1}{2} r \, d\Omega/dr}{\Omega + \frac{1}{2} r \frac{d\Omega}{dr}}$$

< 0

$\Omega \sim 1/r^{\alpha}$



$$= \frac{1}{2} \frac{\cancel{\phi}(-\alpha)}{1 \cancel{\Omega} + \frac{1}{2}(-\alpha)}$$

$\boxed{\alpha < 2}$

$\alpha > 2 \rightarrow$ unstable

$\frac{d}{dr}\left(r^2 \cancel{\Omega}\right) > 0$

< 0

$\ddot{\phi} = -|\,|\, F \rightarrow$ deadlock

eff negative inertial mass.

$\Rightarrow$

stable equilibrium pt.

&

~~turning~~ pt

maximum of ϕ_{eff}.

$\nabla_\phi \Phi$ $\cap \rightarrow$ stable pt.

$\cup$

$F_+ \rightarrow$ decel. exc.

$F_- \rightarrow$ accel exc changes orbit

Donkey Effect

→ L-B i start as donkeys
i.e. if pulled → slow
slowed → accelerate

Now, consider { angular acceleration
tangential force

$$\tau = \frac{dL}{dt} = \frac{Fr}{m}$$

$$\ddot{\phi} = \frac{d\Omega}{dr}\dot{r} = -2A\frac{\dot{r}}{r}$$

but $\dot{r} = \frac{dr}{dL}\dot{L}$

$$= \frac{(Fr)}{m}\, dr/dL$$

$L = \Omega^2 r \qquad dL/dr = 2\Omega r\Omega' + \Omega^2$

$$= -2\Omega B$$

$$\dot{r} = \frac{Fr}{m}(-2\Omega B)^{-1}$$

$$\boxed{r\ddot{\phi} = \frac{AF}{mB}}$$

$$\begin{cases} A = -\frac{1}{2} r\, d\Omega/dr \\ B = -\Omega - \frac{1}{2} r\, d\Omega/dr \end{cases}$$

$B \to \}\ \Omega \sim r^{-\alpha}$ $(\alpha \sim 1)$

$$B = -cr^{-\alpha} + \frac{1}{2}c\alpha r^{-\alpha}$$
$$= -\Omega\left[1 - \alpha/2\right]$$

$$A = -\frac{1}{2} r \frac{d\Omega}{dr} = \frac{c\alpha}{2} r^{-\alpha}$$

$$r\ddot{\phi} = AF/mB = \frac{F}{m}\left[\frac{\alpha/2}{1-\alpha/2}\right]$$

$\alpha < 2 \Rightarrow$ Mass < 0

$\rightarrow$ donkey

Point: Donkey at $\phi_{max} \rightarrow$ stable!

$\Rightarrow$

$$\ddot{r}_1 + \left(\frac{\partial^2 \Phi_0}{\partial r^2} - \Omega_0^2\right)_{r_0} r_1 - 2 r_0 \Omega_0 \dot{\phi}_1$$

$$= -\left(\frac{\partial \Phi_b}{\partial r}\right)_{r_0} \cos\left(m(\Omega_0 - \Omega_b)t\right)$$

$$\ddot{\phi}_1 + 2\Omega_0 \frac{\dot{r}_1}{r_0} = \frac{m \Phi_b(r_0)}{r_0^2} \sin\left[m(\Omega_0 - \Omega_b)t\right]$$

$\Rightarrow$ now, integrate ϕ_1 equation:

$$\dot{\phi}_1 = -2\Omega_0 \frac{r_1}{r} - \frac{\Phi_b(r_0)}{r_0^2(\Omega_0 - \Omega_b)} \cos\left[m(\Omega_0 - \Omega_b)t\right]$$

and plug into r_1 equation $\Rightarrow$

$$\begin{cases} \ddot{r}_1 + \kappa_0^2 r_1 = \left[-\frac{\partial \Phi_b}{\partial r} + \frac{2\Omega_0 \Phi_b}{r(\Omega_0 - \Omega_b)}\right]_{r_0} \cos\left[m(\Omega_0 - \Omega_b)t\right] \\ \kappa_0^2 = \left(\cancel{r} \frac{d\Omega^2}{dr} + 4\Omega^2\right)_{r_0} \end{cases}$$

clearly, resonance if $\begin{cases} m(\Omega_0 - \Omega_b) = \pm \kappa_0 \rightarrow \text{Lindblad resonance} \\ \Omega_0 = \Omega_b \rightarrow \text{corotation resonance} \end{cases}$

Discussion of Resonances:

→ observe, in general, circular orbit has two 'natural frequencies', i.e.

- radial displacement: κ_0
- azimuthal displacement: 0 (i.e. sudden $\Delta\phi$ → particle continues at $\Omega_c - \Omega_b$)

so can understand resonances as

$$\omega_{forcing} = m(\Omega_c - \Omega_b) = \omega_{nat}, \begin{cases} \pm\kappa_0 & \rightarrow \text{Lindblad} \\ 0 & \rightarrow \text{co-rotation} \end{cases}$$

→ physically,

- $\Omega_c = \Omega_b$ → corotation (defines co-rotation radius)
  → guiding center rotates with potential

- $m(\Omega_c - \Omega_b) = \pm\kappa_0$ → Lindblad (defines Lindblad radius)
  → star encounters potential crests at frequency co-inciding with frequency of natural oscillation
  → $+\kappa_0$ → star overtakes potential (inner)
  $-\kappa_0$ → potential overtakes star (outer)

· in hydrodynamic and (i.e. Couette flow)
take $m \neq 0$ but $k_\theta \ll k_r, k_z$

$\Rightarrow$ asymmetry $\Leftrightarrow m \neq 0$

$$(\omega - m\Omega_b)^2 \cong k_z^2 \Phi / (k_r^2 + k_z^2)$$

$\Omega_b \equiv$ fluid rotation speed (no "bar" here)

thus, for external vibration at ω_0

$\omega_0 = m\Omega_b \rightarrow$ co-rotation frequency resonance

$\omega_0 = m\Omega_b \pm \left(\dfrac{k_z^2 \Phi}{k_r^2 + k_z^2} \right)^{1/2} \rightarrow$ Lindblad resonance.
($\sim$ collective mode frequency)

$\Rightarrow$ Near. resonance ?

Recall:

$$\ddot{r}_1 + k_0^2 r_1 = \left[-\frac{\partial \Phi_b}{\partial r} + \frac{2\Omega_0 \Phi_b}{r(\Omega_0 - \Omega_b)} \right]_{r_0} \cos[m(\Omega_0 - \Omega_b)t]$$

clearly, $\rightarrow$ secularity occurs when $r_0 \rightarrow r_{co}, r_{Lind}$.

$\rightarrow$ rescue? – how(? ?) describe orbit.

∴ ① resonances between Ω_b and circular frequency Ω_c
i.e. $\Omega_b = \Omega_c \rightarrow$ co-rotation resonance

② resonance between harmonics of $\Omega_c - \Omega_b$ and epicyclic frequency κ

i.e. $m(\Omega_c - \Omega_b) = \pm \kappa_0 \rightarrow$ Lindblad resonance.

→ Weak Bars – P.T.

- consider $\underline{x}$ in frame rotating at Ω_b, with $\underline{\Omega}_b = \Omega_b \hat{z}$, and r, ϕ in plane ⊥

$$\therefore \left.\frac{d}{dt}\right)_{in} = \left.\frac{d}{dt}\right)_{frm} + \underline{\Omega}_b \times$$

{ i.e. eliminate time dependence via frame change

so

$$\left.\frac{d}{dt}\right)_{frm} = +\left.\frac{d}{dt}\right)_{in} - \underline{\Omega}_b \times$$

Now $\ddot{\underline{r}}_{in} = -\underline{\nabla}\Phi$ Coriolis force   cent. force

$$\underline{\underline{\text{so}}} \quad \ddot{\underline{r}} = -\underline{\nabla}\Phi - 2(\underline{\Omega}_b \times \dot{\underline{r}}) - \underline{\Omega}_b \times (\underline{\Omega}_b \times \underline{r})$$

$$= -\underline{\nabla}\left(\Phi - \frac{1}{2}\Omega_b^2 r^2\right) - 2(\underline{\Omega}_b \times \dot{\underline{r}})$$

centrifugal potential

Observe:

$$\ddot{\underline{r}} = -\underline{\nabla}\left(\Phi - \frac{\Omega_b^2}{2} r^2\right) - 2\left(\underline{\Omega}_b \times \dot{\underline{r}}\right)$$

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} = -\dot{\underline{r}} \cdot \underline{\nabla}\left(\Phi - \Omega_b^2 r^2/2\right)$$

- coriolis force does no work
- can define effective potential

$$\Phi_{eff} = \Phi - \frac{1}{2}\Omega_b^2 r^2$$

so $E_J = \frac{\dot{r}^2}{2} + \Phi_{eff}$; $\frac{dE_J}{dt} = 0$

Jacobi's integral
(Energy with centrifugal potential)

- more generally;

$$E_J = \frac{\dot{r}^2}{2} + \Phi - \frac{1}{2}|\underline{\Omega}_b \times \underline{r}|^2$$

i.e., in polar coordinates:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{\partial \Phi}{\partial r} + 2r\Omega_b\dot{\phi} + \Omega_b^2 r$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} - 2\dot{r}\Omega_b$$

pert.

for weak bar, $\Phi = \Phi_0(r) + \Phi_1(r,\phi)$

perturb about circular orbit

$|\Phi_1/\Phi_0| \ll 1$, small pert.

so, can write similarly:

$$r(t) = r_0 + r_1(t)$$

$$\phi(t) = \phi_0(t) + \phi_1(t)$$

$$\phi_0(t) = (\Omega_0 - \Omega_b)t$$

so, lowest order:

$$\ddot{r}_1 - (r_0+r_1)(\dot{\phi}_0+\dot{\phi}_1)^2 = -\frac{\partial}{\partial r}(\Phi_0+\Phi_1) + 2(r_0+r_1)\Omega_b(\dot{\phi}_0+\dot{\phi}_1) + \Omega_b^2(r_0+r_1)$$

$\Rightarrow$

$$-r_0\dot{\phi}_0^2 = -\frac{\partial \Phi_0}{\partial r} + \Omega_b^2 r_0 + 2r_0\Omega_b\dot{\phi}_0$$

$$\Rightarrow \quad r_0(\dot{\phi}_0+\Omega_b)^2 = \left(\frac{\partial \Phi_0}{\partial r}\right)_{r_0}$$

i.e. radial force balance!

$$r_0\Omega_0^2$$

{defines circular frequency

$$\Omega_c^2(r) = \frac{1}{r}\frac{\partial \Phi_0}{\partial r} \quad \checkmark$$

$$\dot{\phi}_0 = \Omega_c - \Omega_b$$

Similarly, $\phi_0 = (\Omega_c - \Omega_b)t$. transform frame

First Order:

$$\ddot{r}_1 + \left(\frac{\partial^2\Phi_0}{\partial r^2} - \Omega_c^2\right) r_1 - 2r_0\Omega_c\dot{\phi}_1 = -\frac{\partial \Phi_1}{\partial r}\Big|_{r_0}$$

$$\ddot{\phi}_1 + 2\Omega_c\frac{\dot{r}_1}{r_0} = \frac{-1}{r_0^2}\frac{\partial \Phi_1}{\partial \phi}\Big|_{r_0}$$

To make progress:

- fourier analyze Φ_1, i.e.

$$\Phi_1(r,\phi) = \Phi_b(r)\cos m\phi$$

$\phi = \phi_0 + \phi_1$, assume $\phi_1 \ll \phi_0$

$$\Rightarrow \cos(m\phi) \cong \cos(m\phi_0) = \cos(m(\Omega_c - \Omega_b)t)$$

exact

⇒ useful to consider Lagrange points and orbits near them. — i.e. fixed points of co-rotating stationary field
non-pert.

Lagrange points $\begin{cases} \partial\Phi_{eff}/\partial x = 0 \\ \partial\Phi_{eff}/\partial y = 0 \end{cases}$ i.e. equilibrium/stationary (co-rot) points of effective potential.

i.e. at L.P.'s, star travels in circular orbit, appearing stationary in rotating frame ⇒ co-rotation!

Now, consider in-plane (of disk) motion in elliptical potential (rotating) i.e.

$$\Phi_{eff}(x,y) = \Phi_{eff}(x,y,z)\Big|_{z=0} \quad ; \quad E_J = \frac{\dot{r}^2}{2} + \Phi_{eff}$$

$$= \Phi_L(x,y) - \Omega_b^2 r^2$$

where

$$\Phi_L(x,y) = \frac{1}{2} V_0^2 \ln\left(R_c^2 + x^2 + y^2/q^2\right) \quad (q<1).$$

(2D coulomb potential)

asymm.

core radius (inner cutoff)

ellipsoid

→ why $\Phi_L(x,y)$? ⇒ simple non-axisymmetric example!

— equipotentials have constant ratio q ⇒ non-axisymmetry similar at all radii (i.e. captures ellipsoidal shape)

inside

— for $r \ll R_c$ ⇒ expand, yielding

$$\Phi_L(x,y) \cong \frac{V_0^2}{2R_c^2}\left(x^2 + y^2/q^2\right)$$

add const. out.

i.e. Φ is 2D harmonic oscillator, generated by homogeneous ellipsoid $\leftrightarrow$ $R < R_c$, Φ_L approximates homogeneous density distribution (ellipsoidal) potential

[homog. sphere $\rho = \rho_0 \Rightarrow \ddot{r} = -\frac{\partial \phi}{\partial r} = F_r$

$-4\pi r^2 F_r + \frac{4\pi}{3} G \rho_0 r^3 \Rightarrow F_r = -\frac{G\rho_0}{3} r \Rightarrow$ h.o.]

outside

- $R >> R_c$, $q \cong 1 \Rightarrow \phi_L \approx V_0^2 \ln r$

$\Rightarrow$ { $V_c \simeq V_0 (\ln(r))^{1/2} \sim$ const. $\leftrightarrow$ flat rotation curve often observed!

→ why Lagrange points interesting? → Lagrange points are those where star travels on circular orbit, appearing stationary in rotating frame; i.e. $\frac{\partial \phi_{eff}}{\partial x} = 0$, $\frac{\partial \phi_{eff}}{\partial y} = 0$
akin co-rotation

so at L.P.'s:

$$\ddot{\underline{r}} = -2\underline{\Omega} \times \dot{\underline{r}}$$, i.e. only force is coriolis
$\leftrightarrow$ precession.

i.e. clearly L.P.'s $\leftrightarrow$ co-rotation.

$\rightarrow$ L.P.'s for $\Phi_L(x,y)$

Φ_{eff}

$\left\{ \begin{array}{l} \text{B. and T. Fig. 3-13} \\ \text{pg. 137} \\ v_0 = 1,\ q = .8,\ R_c = .1 \\ \Omega_b = 1 \end{array} \right.$

$L_3 \rightarrow$ minima Φ_{eff} Donkey?

$L_4, L_5 \rightarrow$ maxima Φ_{eff}

$L_1, L_2 \rightarrow$ saddle points. Φ_{eff}

"region of corotation" $\equiv$ L_1, L_2 circle
L_4, L_5 circle.

$\rightarrow$ Motion Near L. P.'s?

Expand!
$$\left\{ \begin{array}{l} x = x_L + \varepsilon \\ y = y_L + \eta \end{array} \right.$$

$$\Phi_{eff}(x,y) = \Phi_{eff}(x_L, y_L) + \varepsilon \left.\frac{\partial \Phi_{eff}}{\partial x}\right|_{x_L} + \eta \left.\frac{\partial \Phi_{eff}}{\partial y}\right|_{y_L}$$

$$+ \frac{\varepsilon^2}{2} \left.\frac{\partial^2 \Phi_{eff}}{\partial x^2}\right|_{x_L} + \frac{\eta^2}{2} \left.\frac{\partial^2 \Phi_{eff}}{\partial y^2}\right|_{y_L}$$

$$+ \varepsilon\eta \left.\frac{\partial^2 \Phi_{eff}}{\partial x \partial y}\right|_{x_L, y_L} \rightarrow 0$$

(symmetry $\rightarrow$ bar's principal axis along coord. axis).

$$\Rightarrow \ddot{\xi} = 2\Omega_b \dot{\eta} - \phi_{xx}\xi$$
$$\ddot{\eta} = -2\Omega_b \dot{\xi} - \phi_{yy}\eta$$

equations of motion in vicinity of Lagrange Point.

For stability (i.e. will particle remain at Lagrange point?)

$\xi = x e^{\lambda t}$, $\eta = y e^{\lambda t}$

$$\lambda^2 x = 2\Omega_b \lambda y - \phi_{xx} x$$
$$\lambda^2 y = -2\Omega_b \lambda x - \phi_{yy} y$$

$$\Rightarrow \det \begin{vmatrix} \lambda^2+\phi_{xx}, & -2\lambda y\Omega_b \\ 2\lambda\Omega_b x, & \lambda^2+\phi_{yy} \end{vmatrix} = 0$$

$$\therefore \quad \lambda^4 + \lambda^2(\phi_{xx}+\phi_{yy}+4\Omega_b^2) + \phi_{xx}\phi_{yy} = 0$$

$$\lambda^2 = -\frac{(\quad)}{2} \pm \frac{1}{2}\left[(\quad)^2 - 4\phi_{xx}\phi_{yy}\right]^{1/2}$$

need $\lambda_1^2 < 0$, $\lambda_2^2 < 0$ for stability

Now, for λ^2 real, $\lambda^2 < 0 \Leftrightarrow$ stable L.P.'s:

$$\Rightarrow \begin{cases} \phi_{xx}\phi_{yy} > 0 \\ (\quad) > 2(\phi_{xx}\phi_{yy})^{1/2} \end{cases} \qquad \begin{cases} (\quad) = \phi_{xx}+\phi_{yy} + 4\Omega_b^2 \end{cases}$$

i.e. $\lambda_1^2\lambda_2^2 > 0 \Rightarrow \phi_{xx}\phi_{yy} > 0$

Thus:

$\lambda_1^2 + \lambda_2^2 < 0 \rightarrow -(\phi_{xx}+\phi_{yy}+4\Omega_b^2) < 0$

λ^2 real $\rightarrow (\quad) > 4\phi_{xx}\phi_{yy}$

$\rightarrow$ at saddle points (L_1, L_2) $\phi_{xx}\phi_{yy} < 0$ (i.e. opposite signs) $\Rightarrow$ unstable

$$(\ )$$
$$\Downarrow$$

aside: stability of Lagrange Points for rotating bar model

$$\lambda^4 + \lambda^2(\phi_{xx} + \phi_{yy} + 4\Omega_b^2)$$
$$+ \phi_{xx}\phi_{yy} = 0$$

$\Rightarrow \lambda_1^2$

$\Rightarrow \lambda_2^2$

stability $\left.\begin{matrix}\lambda_1^2 \\ \lambda_2^2\end{matrix}\right\} < 0$

$$\lambda_{1,2}^2 = -\frac{(\ )}{2} \pm \frac{1}{2}\left[(\ )^2 - 4\phi_{xx}\phi_{yy}\right]^{1/2}$$

conditions

① $\lambda_1^2 + \lambda_2^2 = -(\ ) < 0$

$\Downarrow$ sum < 0 if both < 0

② $\lambda_1^2 \lambda_2^2 > 0$ assures both < 0

$\Rightarrow \phi_{xx}\phi_{yy} > 0$

③ λ^2 real $[\ ]^{1/2} > 0$.

$\left.\begin{matrix}L_1 \\ L_2\end{matrix}\right\}$ saddle $\phi_{xx}\phi_{yy} < 0$

$\Rightarrow$ unstable

L_3 $\Rightarrow$ ~~trivial~~ minimum $\rightarrow$ stable, but trivial

$\left.\begin{array}{l} L_4 \\ L_5 \end{array}\right\}$ $\Rightarrow$ maximum $\Rightarrow$ stable modulo parameters.

$\left.\begin{array}{l} L_4 \\ L_5 \end{array}\right\}$ $\Rightarrow$ only potentially relevant Lagrange points.

- clearly, minima stable (L_3)

→ L_4, L_5 stable if exist and $(\;) > 2(\phi_{xx}\phi_{yy})^{1/2}$
i.e. quantitative question. → below

→ For picture of orbit:
have characteristic eqn:

$$\lambda^4 + \lambda^2(\phi_{xx} + \phi_{yy} + 4\Omega_b^2) + \phi_{xx}\phi_{yy} = 0$$

for stable orbits (near L.P.'s), $\lambda^2 = -\alpha^2, -\beta^2$
$\alpha^2, \beta^2 > 0$ i.e

$$\xi = X_1 \cos(\alpha t + \phi_1) + X_2 \cos(\beta t + \phi_2)$$

$$\eta = Y_1 \sin(\alpha t + \phi_1) + Y_2 \sin(\beta t + \phi_2)$$

($0 \leq \alpha \leq \beta$ real)

where orbit eqns ⇒

$$\begin{cases} Y_1 = \dfrac{\phi_{xx} - \alpha^2}{2\Omega_b\alpha} X_1 = \dfrac{2\Omega_b\alpha X_1}{\phi_{yy} - \alpha^2} \\ Y_2 = \dfrac{\phi_{xx} - \beta^2}{2\Omega_b\beta} X_2 = \dfrac{2\Omega_b\beta}{\phi_{yy} - \beta^2} X_2 \end{cases}$$

i.e.

$\Rightarrow$ orbits near L.P.'s are superposition of elliptical motion at $\omega = \alpha, \beta$

$\rightarrow$ for: $\Phi_{eff} = \frac{V_0^2}{2} \ln\left(R_0^2 + x^2 + \frac{y^2}{q^2}\right) - \frac{\Omega_b^2}{2}(x^2+y^2)$

$$e = (1-q^2)^{1/2}, \quad (q<1) \equiv \text{ellipticity}$$

then: L_4, L_5 at $\quad y_L = \left(\frac{V_0^2}{\Omega_b^2} - q^2 R_0^2\right)^{1/2}$

$$x_L = 0$$

$\Rightarrow$ only get L_4, L_5 if $\begin{cases} R_0^2 < V_0^2/q^2\Omega_b^2 \\ \text{or} \\ \Omega_b^2 < V_0^2/q^2 R_0^2 \end{cases}$

Now, crank $\Rightarrow$

$$\phi_{xx}\Big|_{0,y_L} = -\Omega_b^2(1-q^2)$$

$$\phi_{yy}\Big|_{0,y_L} = -2\Omega_b^2\left(1 - q^2(\Omega_b R_0/V_0)^2\right)$$

$$\left(\phi_{xx} + \phi_{yy} + 4\Omega_b^2\right) = \Omega_b^2\left[1 + q^2 + 2q^2\left(\frac{\Omega_b R_0}{V_0}\right)^2\right]$$

i.e. $\phi_{xx}\phi_{yy} > 0$ if L.P.'s exist.

$\Rightarrow$

for $(\ ) > 2(\phi_{xx}\,\phi_{yy})^{1/2}$ (stability), if

(specific case) $R_0 \to 0$, so $(\Omega_b R_0/V_0)^2 \to 0$, then

stability if $\underline{q^2 \gtrsim (.81)^2}$.

In this case:

$$\alpha = e^2\Omega_b^2 = -\Phi_{xx}$$

$$\beta^2 = 2\Omega_b^2$$

P

- α ellipse elongated in x (tangential) direction
- β ellipse: $y_2 = -x_2/\sqrt{2}$ ($\Rightarrow$ retrograde motion!)
  $\beta^2 = 2\Omega_b^2$

$\therefore$

- β ellipse is (basically) epicyclic motion, w/
- α ellipse is slow sloshing in non-axisymmetric ϕ_L. (see 33)

co-rotation

Now, near resonance (original goal):

$$\ddot{r}_1 + \left(\frac{\partial^2 \Phi_0}{\partial r^2} - \Omega^2\right) r_1 - 2 r_0 \Omega_c \dot{\phi}_1 = -\frac{\partial \Phi_1}{\partial r}\Big|_{r_0}$$

$$\ddot{\phi}_1 + 2\Omega_0 \frac{\dot{r}_1}{r_0} = -\frac{1}{r_0^2}\frac{\partial \Phi_1}{\partial \phi}\Big|_{r_0}$$

but! $K^2 = \left(r \frac{\partial}{\partial r}\Omega^2 + 4\Omega^2\right)$ → epi-cyclic frequency

$$r\Omega^2 = \frac{\partial \Phi}{\partial r} \Rightarrow \frac{\partial}{\partial r}(r\Omega^2) = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\cancel{\frac{\partial}{\partial r}(r^2\Omega) = 2r\Omega + r^2\frac{\partial \Omega}{\partial r}} \qquad r\frac{\partial}{\partial r}(\Omega^2) + 4\Omega^2 = \frac{\partial^2 \Phi}{\partial r^2}$$

so

$$\frac{\partial^2 \Phi}{\partial r^2} - \Omega^2 = r\frac{\partial}{\partial r}(\Omega^2) = K^2 - 4\Omega^2$$

$$\Rightarrow \begin{cases} \underset{①}{\ddot{r}_1} + \underset{②}{(K_0^2 - 4\Omega_0^2)\, r_1} - \underset{③}{2 r_0 \Omega_0 \dot{\phi}_1} = \underset{④}{-\frac{\partial \Phi_1}{\partial r}} \\ \ddot{\phi}_1 + 2\Omega_0 \frac{\dot{r}_1}{r_0} = -\frac{1}{r_0^2}\frac{\partial \Phi_1}{\partial \phi} \end{cases}$$

where $\left.\begin{cases} \Omega_0 = \Omega_c(r_0) = \Omega_b \\ \phi_0 = \pi/2 \end{cases}\right\}$ $\underline{\underline{L_5}}$ point $\underline{\underline{\text{max}}}$

$r_0 \equiv$ co-rotation radius.

ordering: ① → e^3

② → e $e \equiv$ ellipticity

③ → e

④ → e^2

e.g. $\Phi_1 \sim e^2$, $r_1 \sim e$ See 33a.

$\frac{d}{dt} \sim e$, $\phi_1 \sim e^0$

so

r_1 equation ⇒

$$2r_0\Omega_0\dot{\phi}_1 = (\kappa_0^2 - 4\Omega_0^2)\, r_1$$

$$\Rightarrow \quad r_1 = \frac{2r_0\Omega_0\dot{\phi}_1}{(\kappa^2 - 4\Omega_0^2)}$$

$$\ddot{\phi}_1 + \frac{4\Omega_0^2}{\kappa_0^2 - 4\Omega_0^2}\ddot{\phi}_1 = -\frac{1}{r_0^2}\frac{\partial \Phi_1}{\partial \phi}\bigg|_{\substack{r_0 \\ \phi_0+\phi_1}}$$

$$\left\{ \frac{\kappa_0^2}{\kappa_0^2 - 4\Omega_0^2}\ddot{\phi}_1 = -\frac{1}{r_0^2}\frac{\partial \Phi_1}{\partial \phi}\bigg|_{\substack{r_0 \\ \phi_0+\phi_1}} \right.$$

$\left(\Phi_1 = \Phi_b(r)\cos(2\phi)\right)$

taking $m=2$ (symmetric bar) ⇒

$$\ddot{\phi}_1 = -\frac{2\Phi_b}{r_0^2}\frac{(4\Omega_0^2 - \kappa_0^2)}{\kappa_0^2}\sin[2(\phi_0+\phi_1)]$$

..2: Ordering

$\Phi_1 \sim e^2 \Rightarrow$ weak non-axisymmetry
$\Leftrightarrow$ assumption

Recall, for L.P. oscillation,

$\alpha^2 = e^2 \Omega_b^2 \Rightarrow \frac{d}{dt} \sim e$

α ellipse tangential $\Rightarrow$ $r_1 \sim e$
$\phi_1 \sim e^0$

this

Now, $\left\{ \begin{array}{l} \alpha^2 = \dfrac{4|\Phi_1|}{r_0^2} \dfrac{(4\Omega_0^2 - \kappa_0^2)}{\kappa_0^2} \\ \psi = 2(\phi_0 + \phi_1) \end{array} \right.$ } related to strength of trapping near co-rotation.

$$\Rightarrow \frac{d^2\psi}{dt^2} = -\alpha^2 \sin\psi \Big\} \rightarrow \text{pendulum!}$$

$$E_\psi = \frac{1}{2}\left(\frac{d\psi}{dt}\right)^2 - \alpha^2 \cos\psi$$

$$= \frac{1}{2}\dot{\psi}^2 + V, \quad V = -\alpha^2 \cos\psi$$

Now:

→ observe $\psi = 2(\phi_0 + \phi_1)$
$= 2\left(\frac{\pi}{2} + \phi_1\right)$, $\phi_1 = 0$ is <u>maximum</u> V

→ $E_\psi < \alpha^2$ → trapped motion (libration) at/around Lagrange point

$E_\psi > \alpha^2$ → rotation/circulation about galactic center.

→ For orbit curve, have

$$r_1 = 2r_0 \Omega_0 \dot{\phi}_1 / (\kappa_0^2 - 4\Omega_0^2)$$

wut $\frac{1}{2}\dot{\phi}_1^2 = E_\psi + \frac{\beta^2}{\alpha^2}\cos\psi$

$\Rightarrow$

$$r_1 = \pm\left(\frac{r_0\Omega_0}{4\Omega_0^2-\kappa_0^2}\right)\left(2\left(E_\psi + \frac{\beta^2}{\alpha^2}\cos(2\phi_1)\right)\right)^{1/2}$$

$\rightarrow$ Contrast:

- previous analysis: valid for small oscillation about L.P. of arbitrary 2D rotating potential

- here, analysis for <u>any</u> amplitude excursion about L_4, L_5 points of weakly non-axisymm. potential.